

# Modified Decision Time of Classical Logic

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[Key Word]: Classical Logic, Gentzen Style, Turing Machine, Decision Time.

## 1 Preliminary

This paper is a modified version of [1]. We cannot calculate all functions even if we restrict its domain to the natural numbers, because the set of all functions is uncountable, and, on the other hand, the set of all computable functions is countable. However, a computable function that we speak of, is not only computable theoretically, but also may spend a million years before we get the answer. The Ackermann function is one of the typical examples. It is computable, but its computing time grows extremely bigger as  $n$  grows bigger. According to our experiment using a computer, we immediately get the answer up to  $n = 3$ , but the computer does not move for  $n = 4$  (of course, it moves underground). It seems that the computing time may spend a million years for  $n = 10$  or  $n = 100$ .

The reading head must scan input data to answer, if the size of the input data is  $n$ , and so the machine needs at least  $n$  steps. Of course, it seems that the machine spends many times according as  $n$  grows. Therefore, it is clear that  $t(n) \geq n$  if  $t(n)$  means the computing time as the function of  $n$ . So, we must argue the  $t(n)$  less than polynomial, or less than exponential, and so on. Now, for the small  $n$ , the computing time is less than some constant, and so we neglect small  $n$ , and argue sufficiently bigger  $n$ .

If  $t(n)$  is the polynomial with degree  $i$ , we can write  $t(n) = a_0n^i + a_1n^{i-1} + \dots$ , but we get  $t(n) < (a_0 + 1)n^i$  for sufficiently bigger  $n$ . So, we write  $t(n) < O(n^i)$  for this fact. We write  $t(n) < O(2^n)$  similarly, in the case of exponential, and so on. Classical logic has two methods, the so-called Hilbert style and Gentzen style, and here we use the Gentzen style. Our logical symbols are  $\neg$ (not),  $\wedge$ (and),  $\vee$ (or),  $\rightarrow$ (if  $\dots$  then  $\dots$ ),  $\forall$ (all), and  $\exists$ (exist).

### Definition of terms:

1. A free variable is a term.
2. If  $f$  is a function symbol and  $a_1, \dots, a_n$  are free variables,  $f(a_1, \dots, a_n)$  is a term.
3. The only terms are those given by 1 to 2.

**Definition of formulas:**

1. If  $P$  is a predicate symbol and  $t_1, \dots, t_n$  are terms,  $P(t_1, \dots, t_n)$  is a formula.
2. If  $A$  and  $B$  are formulas,  $\neg A$ ,  $A \wedge B$ ,  $A \vee B$  and  $A \rightarrow B$  are formulas.
3. If  $a$  is a free variable,  $A(a)$  is a formula, and  $x$  is a bound variable not occurring in  $A(a)$ ,  $\forall x A(x)$  and  $\exists x A(x)$  are formulas.
4. The only formulas are those given by 1 to 3.

If  $\Gamma$  and  $\Delta$  are finite sequences respectively, we call  $\Gamma \vdash \Delta$  a sequent. It means intuitively, that if we assume all of  $\Gamma$  we deduce at least one of  $\Delta$ . If  $A$  is a formula, the sequent  $A \vdash A$  is an axiom. The inference rules are following:

$$\begin{array}{c}
\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \textit{Thinning} \vdash \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \vdash \textit{Thinning} \\
\\
\frac{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta} \textit{Interchange} \vdash \quad \frac{\Gamma \vdash \Delta_1, B, A, \Delta_2}{\Gamma \vdash \Delta_1, A, B, \Delta_2} \vdash \textit{Interchange} \\
\\
\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \textit{Contraction} \vdash \quad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \vdash \textit{Contraction} \\
\\
\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \vdash \neg \quad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} \vdash \neg \\
\\
\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge \vdash \quad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge \vdash \quad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B} \vdash \wedge \\
\\
\frac{A, \Gamma \vdash \Delta \quad B, \Gamma \vdash \Delta}{A \vee B, \Gamma \vdash \Delta} \vee \vdash \quad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vdash \vee \quad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \vdash \vee \\
\\
\frac{\Gamma_1 \vdash \Delta_1, A \quad B, \Gamma_2 \vdash \Delta_2}{A \rightarrow B, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \rightarrow \vdash \quad \frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \rightarrow B} \vdash \rightarrow \\
\\
\frac{A(t), \Gamma \vdash \Delta}{\forall x A(x), \Gamma \vdash \Delta} \forall \vdash \quad \frac{\Gamma \vdash \Delta, A(a)}{\Gamma \vdash \Delta, \forall x A(x)} \vdash \forall \\
\\
\frac{A(a), \Gamma \vdash \Delta}{\exists x A(x), \Gamma \vdash \Delta} \exists \vdash \quad \frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x A(x)} \vdash \exists \\
\\
\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \textit{cut}
\end{array}$$

where  $t$  in the inference rule of  $\forall$  and  $\exists$  is any term, and  $a$  is any free variable not occurring lower sequent.

## 2 Main Theorem

Provability is equivalent to provability without cut, and so it is sufficient to argue the decision of the provability without cut. In the proof figure without cut, we attend that the logical symbols including upper sequent are fewer than those including the lower sequent. We neglect  $\forall$  and  $\exists$  during short time, because these symbols are complicated. Let  $n$  be the length of a given sequent. We will show that we decide whether the sequent is provable or not within  $O(2^{n \log n})$  steps (in the case of propositional logic, within  $O(2^n)$  steps), by using induction on  $n$ . (This is the time to spend for the decision of provability of the sequent, but not the time to prove it.)

**Case 1:** No logical symbol occurring in  $\Gamma \vdash \Delta$ .

In the case that  $\Gamma$  and  $\Delta$  have the same formula, it is provable, and in the other case, it is unprovable. We can decide it within  $O(n^2) \leq O(2^n)$  steps.

**Case 2:**  $\Gamma_1, \neg A, \Gamma_2 \vdash \Delta$ .

Provability of this is equivalent to one of  $\Gamma_1, \Gamma_2 \vdash \Delta, A$ . The length of  $\Gamma_1, \Gamma_2 \vdash \Delta, A$  is less than  $n$ . So, the provability of this can be decided within  $O(2^{n-1})$  steps, by hypothesis of induction. And, we can decide whether it is the form  $\Gamma_1, \neg A, \Gamma_2 \vdash \Delta$  or not within  $O(n)$  steps. So, provability of this sequent can be decided within  $O(n) + O(2^{n-1}) \leq O(2^n)$  steps.

**Case 3:**  $\Gamma \vdash \Delta_1, \neg A, \Delta_2$ .

similarly.

**Case 4:**  $\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta$ .

Provability of this is equivalent one of  $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$ . Because if  $\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta$  is provable,

(i)  $\Gamma_1, A, \Gamma_2 \vdash \Delta$  is provable

or

(ii)  $\Gamma_1, B, \Gamma_2 \vdash \Delta$  is provable.

In the case of (i), the proof figure of this sequent is of the form

$$\frac{\frac{A \vdash A \quad B \vdash B \quad C \vdash C}{\vdots} \quad \vdots}{\Gamma_1, A, \Gamma_2 \vdash \Delta}.$$

Then,

$$\frac{\frac{A \vdash A}{B, A \vdash A} \quad \frac{B \vdash B}{B, B \vdash B} \quad \frac{C \vdash C}{B, C \vdash C}}{\vdots} \quad \vdots}{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}$$

is a proof figure of  $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$ .

Conversely, if there is a proof figure of  $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$

is a proof figure of  $\Gamma_1, A \wedge B, \Gamma_2 \vdash \Delta$ . Now, if we consider that  $A \wedge B$  is the abbreviation of  $(A \wedge B)$ , the length of  $\Gamma_1, A, B, \Gamma_2 \vdash \Delta$  is less than  $n$ . The remainder is similar to Case 2.

**Case 5:**  $\Gamma \vdash \Delta_1, A \wedge B, \Delta_2$ .

Provability of this is equivalent to provability of both of  $\Gamma \vdash \Delta_1, A, \Delta_2$  and  $\Gamma \vdash \Delta_1, B, \Delta_2$ . The decision of the former is of  $O(n) + O(2^{n-1})$ , and one of the latter  $O(n) + O(2^{n-1})$ , and therefore the decision of  $\Gamma \vdash \Delta_1, A \wedge B, \Delta_2$  is of  $O(n) + O(2^{n-1}) + O(n) + O(2^{n-1}) \leq O(2^n)$  as a whole.

**Case 6:**  $\Gamma_1, A \vee B, \Gamma_2 \vdash \Delta$ .

Similarly to Case 5.

**Case 7:**  $\Gamma \vdash \Delta_1, A \vee B, \Delta_2$ .

Similarly to Case 4.

**Case 8:**  $\Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta$ .

Provability of this is equivalent to provability of both of  $\Gamma_1, \Gamma_2 \vdash \Delta, A$  and  $\Gamma_1, B, \Gamma_2 \vdash \Delta$ . Because that if  $\Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta$  is provable, there is  $\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}, \Delta_1, \Delta_2$  such that  $\Gamma_{11} \subseteq \Gamma_1, \Gamma_{12} = \Gamma_1 - \Gamma_{11}, \Gamma_{21} \subseteq \Gamma_2, \Gamma_{22} = \Gamma_2 - \Gamma_{21}, \Delta_1 \subseteq \Delta, \Delta_2 = \Delta - \Delta_1$ , and further  $\Gamma_{11}, \Gamma_{21} \vdash \Delta_1, A$  and  $\Gamma_{12}, B, \Gamma_{22} \vdash \Delta_2$  are provable. If

$$\frac{\frac{A \vdash A}{\vdots} \quad \frac{B \vdash B}{\vdots} \quad \frac{C \vdash C}{\vdots}}{\Gamma_{11}, \Gamma_{21} \vdash \Delta_1, A}$$

is a proof figure of  $\Gamma_{11}, \Gamma_{21} \vdash \Delta_1, A$ , then

$$\frac{\frac{A \vdash A}{\Gamma_{12}, A, \Gamma_{22} \vdash \Delta_2, A} \quad \frac{B \vdash B}{\Gamma_{12}, B, \Gamma_{22} \vdash \Delta_2, B} \quad \frac{C \vdash C}{\Gamma_{12}, C, \Gamma_{22} \vdash \Delta_2, C}}{\Gamma_1, \Gamma_2 \vdash \Delta, A}$$

is a proof figure of  $\Gamma_1, \Gamma_2 \vdash \Delta, A$ .

Similarly, if  $\Gamma_{11}, B, \Gamma_{21} \vdash \Delta_1$  is provable, then  $\Gamma_1, B, \Gamma_2 \vdash \Delta$  is also provable.

Conversely, if  $\Gamma_1, \Gamma_2 \vdash \Delta, A$  and  $\Gamma_1, B, \Gamma_2 \vdash \Delta$  is provable, then

$$\frac{\frac{\frac{\vdots}{\Gamma_1, \Gamma_2 \vdash \Delta, A} \quad \frac{\vdots}{\Gamma_1, B, \Gamma_2 \vdash \Delta}}{\Gamma_1, \Gamma_1, A \rightarrow B, \Gamma_2, \Gamma_2 \vdash \Delta, \Delta}}{\Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta}$$

is a proof figure of  $\Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta$ .

$\Gamma_1, \Gamma_2 \vdash \Delta, A$  and  $\Gamma_1, B, \Gamma_2 \vdash \Delta$  can be decided  $O(2^{n-1})$  steps, respectively, and hence  $\Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta$  is decided within  $O(n) + O(2^{n-1}) + O(n) + O(2^{n-1}) \leq O(2^n)$  steps, wholly.

**Case 9:**  $\Gamma \vdash \Delta_1, A \rightarrow B, \Delta_2$ .

Provability of this is equivalent to provability of  $A, \Gamma \vdash \Delta_1, B, \Delta_2$ , and the latter can be decided within  $O(2^{n-1})$  steps, and therefore the former can be decided within  $O(n) + O(2^{n-1}) \leq O(2^n)$  steps.

Unless  $\forall$  or  $\exists$ , we get the same result in spite of an order of elimination of logical symbols. But we will fail to prove a provable sequent, if we use a wrong order of eliminations. This comes from the very strong condition that  $a$  in  $\vdash \forall$  or  $\exists \vdash$  must be a free variable not occurring in the lower sequent.

We select the most alternating formula of  $\forall$  and  $\exists$  for a given sequent. On the occasion, we count alternated quantifiers for  $\forall$  or  $\exists$  in  $\neg$ , or in the leftside of  $\rightarrow$ . Last, we alternate all quantifiers of the leftside of  $\vdash$ , again. In the case that most alternating formulas are plural, we select the formula starting from  $\forall$ . In the case that such formulas are yet plural, we may select any formula. The time for selecting such a formula can be within  $O(n^2)$  steps.

**Case 10:**  $\Gamma_1, \forall x A(x), \Gamma_2 \vdash \Delta$ .

The provability of this is equivalent to provability of  $\Gamma_1, A(t), \Gamma_2 \vdash \Delta$  for some term  $t$ . But, the latter may not be shorter than the former contrary to the cases so far dealt with. So, in this case we devise as follows. We notice no inference rule to change contents of the term  $t$  in general predicate logic. (This argument does not good in a special predicate logic, for example, the natural number theory.) Namely, when a term  $t$  occur anywhere in the proof figure, corresponding part remain  $t$  in all sequent over that place. Therefore, the figure which replaced the term  $t$  occurring in the upper sequent of  $\forall \vdash$  or  $\vdash \exists$  in anywhere in the proof figure, to a free variable  $a$  not occurring in the lower sequent throughout whole the proof figure, is yet a proof figure. For example, the leftside figure is before replacing and the rightside figure is after replacing, in the following examples:

$$\frac{P(f(c)) \vdash P(f(c))}{\forall x P(x) \vdash P(f(c))} \quad \frac{P(a) \vdash P(a)}{\forall x P(x) \vdash P(a)}$$

$$\frac{P(f(c)) \vdash P(f(c))}{\forall x P(f(x)) \vdash P(f(c))} \quad \frac{P(f(a)) \vdash P(f(a))}{\forall x P(f(x)) \vdash P(f(a))}$$

The provability of  $\Gamma_1(t), \forall x A(x, t), \Gamma_2(t) \vdash \Delta(t)$  is equivalent to one of

$$\Gamma_1(t), A(t, t), \Gamma_2(t) \vdash \Delta(t)$$

and moreover this is equivalent to one of

$$\Gamma_1(a), A(a, a), \Gamma_2(a) \vdash \Delta(a).$$

And the last sequent is shorter than the first sequent. Therefore, the last sequent can be decided within  $O(2^{(n-1)\log(n-1)})$  steps. And, the selections of  $t$  are less than  $n$ , and so the first sequent can be decided within  $O(n) + O(n^2) + O((n-1) \cdot 2^{(n-1)\log(n-1)}) \leq O(2^{n\log n})$  steps.

**Case 11:**  $\Gamma \vdash \Delta_1, \forall x A(x), \Delta_2$ .

The provability of this is equivalent to one of  $\Gamma \vdash \Delta_1, A(a), \Delta_2$  for a free variable  $a$  not occurring in the sequent. And since the length of the latter is shorter than one of the former, it can be decided within  $O(n) + O(n^2) + O(2^{(n-1)\log(n-1)}) \leq O(2^{n\log n})$  steps.

**Case 12:**  $\Gamma_1, \exists x A(x), \Gamma_2 \vdash \Delta$ .

Similarly to Case 11.

**Case 13:**  $\Gamma \vdash \Delta_1, \exists x A(x), \Delta_2$ .

Similarly to Case 10.

From the above results we can get the following theorem.

**Main Theorem 1** *In the classical logic, we can decide that  $\Gamma \vdash \Delta$  is provable or not, within  $O(2^{n\log n})$  steps (in the case of propositional logic, within  $O(2^n)$  steps).*

### 3 Conclusion

In this paper, we argue by Gentzen style, but it is well known that the provability by Gentzen style and the one by Hilbert style are equivalent. On the other hand, it is well known that the provability of a sequent and the validity of it are equivalent. Therefore, the decision time of provability and one of validity are equal. In the propositional logic, it is trivial that validity of a sequent can be decided within exponential time, since it is valid if it is true for all combinations of the truth values of the atomic formulas contained in the sequent. But, in the predicate logic, it is impossible that we decide the truth value of  $\forall x A(x)$  or  $\exists x A(x)$  within finite times. It is due that we must examine the truth values of  $A(t)$  for all terms  $t$ . In this paper, we showed that it is sufficient within  $O(2^{n\log n})$  steps (in the case of propositional logic,  $O(2^n)$  steps), but not necessary within them. It may be decided within polynomial times. It yet remains to show its impossibility within polynomial times. We began with the most easy classical logic at the start, and we also will examine the other non-classical logics.

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### **[References]**

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## 古典論理の決定時間 (改訂版)

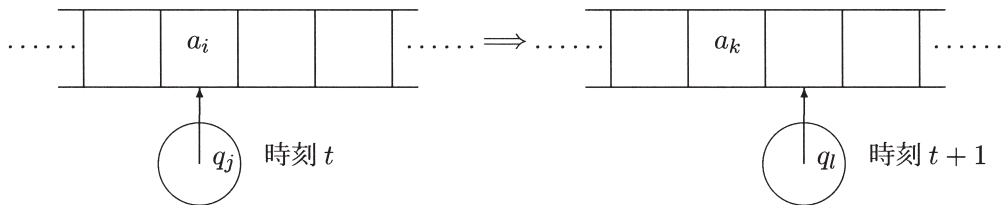
### [要旨]

論理学における sequent とはある法則に従って並べられた文字列の事である。ここで許される文字は自由変数 ( $a_0, a_1, a_2, \dots$ ), 束縛変数 ( $x_0, x_1, x_2, \dots$ ), 関数記号 ( $f_0, f_1, f_2, \dots$ ), 述語記号 ( $P_0, P_1, P_2, \dots$ ), 論理記号 ( $\neg, \wedge, \vee, \rightarrow, \forall, \exists, \vdash$ ) 補助記号 (カッコとコンマ) だけである。term と formula は以下のように帰納的に定義される:

1. 自由変数は term である。
2.  $f$  が関数記号で,  $t_1, \dots, t_n$  が term のとき,  $f(t_1, \dots, t_n)$  は term である。
3.  $P$  が述語記号で,  $t_1, \dots, t_n$  が term のとき,  $P(t_1, \dots, t_n)$  は formula である。
4.  $A, B$  が formula のとき,  $\neg A, A \wedge B, A \vee B, A \rightarrow B$  は formula である。
5.  $A(a)$  が自由変数  $a$  を含む formula で,  $x$  が  $A(a)$  の中に入らない束縛変数のとき,  $\forall x A(x), \exists x A(x)$  は formula である。
6. 1 ~ 5 以外の方法では term や formula は得られない。

sequent は「formula の有限列  $\vdash$  formula の有限列」という形の有限列の事である。sequent の証明と言うのは公理から出発して推論規則のみを用いて導き出す事である。(古典論理の公理と推論規則は本文を参照されたい。)

Turing machine とはセルによって区切られた左右に無限に長いテープと読み取りヘッドと有制御部からなる機械で, 以下の条件を満たす理論上のデジタルコンピュータのことである。テープ上の各セルには1つのセルに1文字だけ書かれている。(空白文字も特殊文字の1つとみなす。) 読み取りヘッドは各時点で1つのセルだけを見ている。各時点で, 読み取りヘッドはセルから読み取った文字とその時点における有制御部の内部状態との組み合わせによってその文字を書き換え, 右隣または左隣のセルに移動し, 内部状態を変化させ, 次の時点へと移る。



Turing machine に文字列を入力してから出力が得られるまでの時間は一般に入力が大きくなるほど時間を消費するので, その時間を入力文字列の長さ  $n$  の関数で表す。もちろん, Turing machine の性能によって消費時間は異なる。例えば昔のコンピュータは遅いが最近のコンピュータは速い。しかし, この時間の違いは高々定数倍である。従って, 定数倍の違いは無視する。すなわち性能による差は無視して, もっと本



質的な処理時間について考える．また， $n$  が比較的小さい場合は定数で押さえられるから小さい  $n$  は無視して，十分大きい  $n$  について考える．例えば，処理時間が多項式  $a_0n^i + a_1n^{i-1} + \cdots + a_{i-1}n + a_i$  の場合十分大きい  $n$  に対してはこの式は  $(a_0 + 1)n^i$  で押さえられ， $a_0 + 1$  は定数だからこれを  $n^i$  と同一視する．

古典論理において与えられた sequent が証明可能か否かの判定は時間をいとわなければ可能である事はよく知られている．しかしながら，sequent の証明可能性の判定に要する時間はどのくらい必要かということはあまり研究されていないようである．そこで，その判定時間を調べる事は興味深い事と思われる．本論分では  $2^{n \log n}$  ステップで (命題論理の場合は  $2^n$  ステップで) これが可能である事を示した．(古典論理というのは古典述語論理の事であり，特に  $\forall$  と  $\exists$  を用いない論理を古典命題論理という．)